Lecture 39

Quantum Coherent State of Light

The Schrödinger wave equation was a great success. When it was discovered by Schrödinger, and appplied to a very simple hydrogen atom, its eigensolutions, especially the eigenvalues E_n coincide beautifully with spectroscopy experiment of the hydrogen atom. Since the electron wavefunctions inside a hydrogen atom does not have a classical analog, less was known about these wavefunctions. But in QED and quantum optics, the wavefunctions have to be connected with classical electromagnetic oscillations. As seen previously, electromagnetic oscillations resemble those of a pendulum. The original eigenstates of the quantum pendulum were the photon number states also called the Fock states. The connection to the classical pendulum was tenuous, but required by the correspondence principle–quantum phenomena resembles classical phenomena in the high energy limit. This connection was finally established by the establishment of the coherent state.

39.1 The Quantum Coherent State

We have seen that a photon number states[1] of a quantum pendulum do not have a classical correspondence as the average or expectation values of the position and momentum of the pendulum are always zero for all time for this state. Therefore, we have to seek a time-dependent quantum state that has the classical equivalence of a pendulum. This is the coherent state, which is the contribution of many researchers, most notably, Roy Glauber (1925–2018) [253] in 1963, and George Sudarshan (1931–2018) [254]. Glauber was awarded the Nobel prize in 2005.

We like to emphasize again that the modes of an electromagnetic cavity oscillation are homomorphic to the oscillation of classical pendulum. Hence, we first connect the oscillation of a quantum pendulum to a classical pendulum. Then we can connect the oscillation of

[1]In quantum theory, a "state" is synonymous with a state vector or a function.

a quantum electromagnetic mode to the classical electromagnetic mode and then to the quantum pendulum.

39.1.1 Quantum Harmonic Oscillator Revisited

To this end, we revisit the quantum harmonic oscillator or the quantum pendulum with more mathematical depth. Rewriting Schrödinger equation as the eigenequation for the photon number state for the quantum harmonic oscillator, we have

$$\hat{H}\psi_n(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\right]\psi_n(x) = E_n\psi_n(x). \tag{39.1.1}$$

where $\psi_n(x)$ is the eigenfunction, and E_n is the eigenvalue. The above can be changed into a dimensionless form first by dividing $\hbar\omega_0$, and then let $\xi = \sqrt{\frac{m\omega_0}{\hbar}}x$ be a dimensionless variable. The above then becomes

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right)\psi(\xi) = \frac{E}{\hbar\omega_0}\psi(\xi) \tag{39.1.2}$$

We can define $\hat{\pi} = -i\frac{d}{d\xi}$ and $\hat{\xi} = \hat{I}\xi$ to rewrite the Hamiltonian as

$$\hat{H} = \frac{1}{2}\hbar\omega_0(\hat{\pi}^2 + \hat{\xi}^2) \tag{39.1.3}$$

Furthermore, the Hamiltonian in (39.1.2) looks almost like $A^2 - B^2$, and hence motivates its factorization. To this end, we first show that

$$\frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) = \frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right) - \frac{1}{2}\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right) \tag{39.1.4}$$

It can be shown easily that as operators (meaning that they will act on a function to their right), the last term on the right-hand side is an identity operator, namely that

$$\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right) = \hat{I} \tag{39.1.5}$$

Therefore

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right) = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) + \frac{1}{2} \tag{39.1.6}$$

We define the operator

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \tag{39.1.7}$$

The above is the creations, or raising operator and the reason for its name is obviated later. Moreover, we define

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) \tag{39.1.8}$$

which represents the annihilation or lowering operator. With the above definitions of the raising and lowering operators, it is easy to show that by straightforward substitution that

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I} \tag{39.1.9}$$

Therefore, Schrödinger equation (39.1.2) for quantum harmonic oscillator can be rewritten more concisely as

$$\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)\psi = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\psi = \frac{E}{\hbar\omega_0}\psi \tag{39.1.10}$$

In mathematics, a function is analogous to a vector. So ψ is the implicit representation of a vector. The operator

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

is an implicit[2] representation of an operator, and in this case, a differential operator. So in the above, (39.1.10), is analogous to the matrix eigenvalue equation $\overline{\mathbf{A}} \cdot \mathbf{x} = \lambda\mathbf{x}$.

Consequently, the Hamiltonian operator can be expressed concisely as

$$\hat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{39.1.11}$$

Equation (39.1.10) above is in implicit math notation. In implicit Dirac notation, it is

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\psi\rangle = \frac{E}{\hbar\omega_0}|\psi\rangle \tag{39.1.12}$$

In the above, $\psi(\xi)$ is a function which is a vector in a functional space. It is denoted as ψ in math notation and $|\psi\rangle$ in Dirac notation. This is also known as the "ket". The conjugate transpose of a vector in Dirac notation is called a "bra" which is denoted as $\langle\psi|$. Hence, the inner product between two vectors is denoted as $\langle\psi_1|\psi_2\rangle$ in Dirac notation.[3]

If we denote a photon number state by $\psi_n(x)$ in explicit notation, ψ_n in math notation or $|\psi_n\rangle$ in Dirac notation, then we have

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\psi_n\rangle = \frac{E_n}{\hbar\omega_0}|\psi_n\rangle = \left(n + \frac{1}{2}\right)|\psi_n\rangle \tag{39.1.13}$$

where we have used the fact that $E_n = (n + 1/2)\hbar\omega_0$. Therefore, by comparing terms in the above, we have

$$\hat{a}^\dagger\hat{a}|\psi_n\rangle = n|\psi_n\rangle \tag{39.1.14}$$

and the operator $\hat{a}^\dagger\hat{a}$ is also known as the number operator because of the above. It is often denoted as

$$\hat{n} = \hat{a}^\dagger\hat{a} \tag{39.1.15}$$

[2]A notation like $\overline{\mathbf{A}} \cdot \mathbf{x}$, we will call implicit, while a notation $\sum_{i,j} A_{ij}x_j$, we will call explicit.

[3]There is a one-to-one correspondence of Dirac notation to matrix algebra notation. $\hat{A}|x\rangle \leftrightarrow \overline{\mathbf{A}}\cdot\mathbf{x}$, $\langle x| \leftrightarrow \mathbf{x}^\dagger$ $\langle x_1|x_2\rangle \leftrightarrow \mathbf{x}_1^\dagger \cdot \mathbf{x}_2$.

and $|\psi_n\rangle$ is an eigenvector of $\hat{n} = \hat{a}^\dagger \hat{a}$ operator with eigenvalue n. It can be further shown by direct substitution that

$$\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle \quad \Leftrightarrow \hat{a}|n\rangle \; = \sqrt{n}|n-1\rangle \tag{39.1.16}$$

$$\hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle \Leftrightarrow \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{39.1.17}$$

hence their names as lowering and raising operator.[4]

39.2 Some Words on Quantum Randomness and Quantum Observables

We saw previously that in classical mechanics, the conjugate variables p and x are deterministic variables. But in the quantum world, they become random variables with means and variance. It was quite easy to see that x is a random variable in the quantum world. But the momentum p is elevated to become a differential operator $\hat{p}$, and it is not clear that it is a random variable anymore.

Quantum theory is a lot richer in content than classical theory. Hence, in quantum theory, conjugate variables like p and x are observables endowed with the properties of mean and variance. For them to be endowed with these properties, they are elevated to become quantum operators, which are the representations of these observables. To be meaningful, a quantum state $|\psi\rangle$ has to be defined for a quantum system, and these operators represent observables act on the quantum state.

Henceforth, we have to extend the concept of the average of a random variable to the "average" of a quantum operator. Now that we know Dirac notation, we can write the expectation value of the operator $\hat{p}$ with respect to a quantum state ψ as

$$\langle \hat{p} \rangle = \langle \psi | \hat{p} | \psi \rangle = \bar{p} \tag{39.2.1}$$

The above is the elevated way of taking the "average" of an operator which is related to the mean of the random variable p.

As mentioned before, Dirac notation is homomorphic to matrix algebra notation. The above is similar to $\boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{P}} \cdot \boldsymbol{\psi} = \bar{p}$. This quantity $\bar{p}$ is always real if $\overline{\mathbf{P}}$ is a Hermitian matrix. Hence, in (39.2.1), the expectation value $\bar{p}$ is always real if $\hat{p}$ is Hermitian. In fact, it can be proved that $\hat{p}$ is Hermitian in the function space that it is defined.

Furthermore, the variance of the random variable p can be derived from the quantum operator $\hat{p}$ with respect to to a quantum state $|\psi\rangle$. It is defined as

$$\sigma_p^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \tag{39.2.2}$$

where σ_p is the standard deviation of the random variable p and σ_p^2 is its variance [64, 65].

The above implies that the definition of the quantum operators and the quantum states is not unique. One can define a unitary matrix or operator $\overline{\mathbf{U}}$ such that $\overline{\mathbf{U}}^\dagger \cdot \overline{\mathbf{U}} = \overline{\mathbf{I}}$. Then

[4]The above notation for a vector could appear cryptic or too terse to the uninitiated. To parse it, one can always down-convert from an abstract notation to a more explicit notation. Namely, $|n\rangle \to |\psi_n\rangle \to \psi_n(\xi)$.

the new quantum state is now given by the unitary transform $\boldsymbol{\psi}' = \overline{\mathbf{U}} \cdot \boldsymbol{\psi}$. With this, we can easily show that

$$\bar{p} = \boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{P}} \cdot \boldsymbol{\psi} = \boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{U}}^\dagger \cdot \overline{\mathbf{U}} \cdot \overline{\mathbf{P}} \cdot \overline{\mathbf{U}}^\dagger \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\psi}$$
$$= \boldsymbol{\psi}'^\dagger \cdot \overline{\mathbf{P}}' \cdot \boldsymbol{\psi}' \tag{39.2.3}$$

where $\overline{\mathbf{P}}' = \overline{\mathbf{U}} \cdot \overline{\mathbf{P}} \cdot \overline{\mathbf{U}}^\dagger$ via unitary transform. Now, $\overline{\mathbf{P}}'$ is the new quantum operator representing the observable p and $\boldsymbol{\psi}'$ is the new quantum state vector.

In the previous section, we have elevated the position variable or observable ξ to become an operator $\hat{\xi} = \xi \hat{I}$. This operator is clearly Hermitian, and hence, the expectation value of this position operator is always real. Here, $\hat{\xi}$ is diagonal in the coordinate space representation, but it need not be in other Hilbert space representations.

39.3 Derivation of the Coherent States

As one cannot see the characteristics of a classical pendulum emerging from the photon number states, one needs another way of bridging the quantum world with the classical world. This is the role of the coherent state: It will show the correspondence principle, with a classical pendulum emerging from a quantum pendulum when the energy of the pendulum is large. Hence, it will be interesting to see how the coherent state is derived.

The derivation of the coherent state is more math than physics. Nevertheless, the derivation is interesting. We are going to present it according to the simplest way presented in the literature. There are deeper mathematical methods to derive this coherent state like Bogoliubov transform which is outside the scope of this course.

Now, endowed with the needed mathematical tools, we can derive the coherent state. To say succinctly, the coherent state is the eigenstate of the annihilation operator, namely that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{39.3.1}$$

Here, we use α as an eigenvalue as well as an index or identifier of the state $|\alpha\rangle$.[5] Since the number state $|n\rangle$ is complete, the coherent state $|\alpha\rangle$ can be expanded in terms of the number state $|n\rangle$. Or that

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{39.3.2}$$

When the annihilation operator is applied to the above, we have

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} C_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n \sqrt{n}|n-1\rangle$$
$$= \sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle \tag{39.3.3}$$

[5]This notation is cryptic and terse, but one can always down-convert it as $|\alpha\rangle \rightarrow |f_\alpha\rangle \rightarrow f_\alpha(\xi)$ to get a more explicit notation.

Equating the above with $\alpha|\alpha\rangle$ on the right-hand side of (39.3.1), then

$$\sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle = \alpha \sum_{n=0}^{\infty} C_n|n\rangle \tag{39.3.4}$$

By the orthonormality of the number states $|n\rangle$ and the completeness of the set,

$$C_{n+1} = \alpha C_n/\sqrt{n+1} \tag{39.3.5}$$

Or recursively

$$C_n = C_{n-1}\alpha/\sqrt{n} = C_{n-2}\alpha^2/\sqrt{n(n-1)} = \ldots = C_0\alpha^n/\sqrt{n!} \tag{39.3.6}$$

Consequently, the coherent state $|\alpha\rangle$ is

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{39.3.7}$$

But due to the probabilistic interpretation of quantum mechanics, the state vector $|\alpha\rangle$ is normalized to one, or that[6]

$$\langle\alpha|\alpha\rangle = 1 \tag{39.3.8}$$

Then

$$\begin{aligned}\langle\alpha|\alpha\rangle &= C_0^* C_0 \sum_{n,n'}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{n'}}{\sqrt{n'!}} \langle n'|n\rangle \\ &= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2} = 1\end{aligned} \tag{39.3.9}$$

Therefore, $C_0 = e^{-|\alpha|^2/2}$, or that

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{39.3.10}$$

In the above, to reduce the double summations into a single summation, we have made use of $\langle n'|n\rangle = \delta_{n'n}$, or that the photon-number states are orthonormal. Also since $\hat{a}$ is not a Hermitian operator, its eigenvalue α can be a complex number.

Since the coherent state is a linear superposition of the photon number states, an average number of photons can be associated with the coherent state. If the average number of photons embedded in a coherent is N, then it can be shown that $N = |\alpha|^2$. As shall be shown, α is related to the amplitude of the quantum oscillation: The more photons there are, the larger is $|\alpha|$.

[6]The expression can be written more explicitly as $\langle\alpha|\alpha\rangle = \langle f_\alpha|f_\alpha\rangle = \int_{-\infty}^{\infty} d\xi f_\alpha^*(\xi) f_\alpha(\xi) = 1$.

39.3.1 Time Evolution of a Quantum State

The Schrödinger equation can be written concisely as

$$\hat{H}|\psi\rangle = i\hbar\partial_t|\psi\rangle \tag{39.3.11}$$

The above not only entails the form of Schrödinger equation, it is the form of the general quantum state equation. Since $\hat{H}$ is time independent, the formal solution to the above is

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle \tag{39.3.12}$$

Applying this to the photon number state with $\hat{H}$ being that of the quantum pendulum, then

$$e^{-i\hat{H}t/\hbar}|n\rangle = e^{-i\omega_n t}|n\rangle \tag{39.3.13}$$

where $\omega_n = \left(n+\frac{1}{2}\right)\omega_0$. The above simplification follows because $|n\rangle$ an eigenstate of the Hamiltonian $\hat{H}$ for the quantum pendulum, or that

$$\hat{H}|n\rangle = \hbar\omega_n|n\rangle = \hbar\omega_0\left(n+\frac{1}{2}\right)|n\rangle \tag{39.3.14}$$

In other words, $|n\rangle$ is an eigenvector of $\hat{H}$.

Time Evolution of the Coherent State

Using the above time-evolution operator, then the time dependent coherent state evolves in time as[7]

$$|\alpha, t\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\omega_n t}}{\sqrt{n!}}|n\rangle \tag{39.3.15}$$

By letting $\omega_n = \omega_0\left(n+\frac{1}{2}\right)$, the above can be written as

$$|\alpha, t\rangle = e^{-i\omega_0 t/2}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega_0 t}\right)^n}{\sqrt{n!}}|n\rangle \tag{39.3.16}$$

Now we see that the last factor in (39.3.16) is similar to the expression for a coherent state in (39.3.10). Therefore, we can express the above more succinctly by replacing α in (39.3.10) with $\tilde{\alpha} = \alpha e^{-i\omega_0 t}$ as

$$|\alpha, t\rangle = e^{-i\omega_0 t/2}|\alpha e^{-i\omega_0 t}\rangle = e^{-i\omega_0 t/2}|\tilde{\alpha}\rangle \tag{39.3.17}$$

Consequently,

$$\hat{a}|\alpha, t\rangle = e^{-i\omega_0 t/2}\left(\alpha e^{-i\omega_0 t}\right)|\alpha e^{-i\omega_0 t}\rangle = \tilde{\alpha}|\alpha, t\rangle \tag{39.3.18}$$

Therefore, $|\alpha, t\rangle$ is the eigenfunction of the $\hat{a}$ operator with eigenvalue $\tilde{\alpha}$. But now, the eigenvalue of the annihilation operator $\hat{a}$ is a complex number which is a function of time t. It is to be noted that in the coherent state, the photon number states time evolve coherently together in a manner to result in a phase shift $e^{-i\omega_0 t}$ in the eigenvalue!

[7] Note that $|\alpha, t\rangle$ is a shorthand for $f_\alpha(\xi, t)$.

39.4 More on the Creation and Annihilation Operator

In order to connect the quantum pendulum to a classical pendulum via the coherent state, we will introduce some new operators. Since

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \tag{39.4.1}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) \tag{39.4.2}$$

We can relate $\hat{a}^\dagger$ and $\hat{a}$, which are non-hermitian, to the momentum operator $\hat{\pi}$ and position operator $\hat{\xi}$ previously defined which are hermitian. Then

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-i\hat{\pi} + \hat{\xi}\right) \tag{39.4.3}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(i\hat{\pi} + \hat{\xi}\right) \tag{39.4.4}$$

We also notice that

$$\hat{\xi} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) = \xi\hat{I} \tag{39.4.5}$$

$$\hat{\pi} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right) = -i\frac{d}{d\xi} \tag{39.4.6}$$

Notice that both $\hat{\xi}$ and $\hat{\pi}$ are Hermitian operators in the above, with real expectation values. With this, the average or expectation value of the position of the pendulum in normalized coordinate, ξ, can be found by taking expectation with respect to the coherent state, or

$$\langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\hat{a}^\dagger + \hat{a}|\alpha\rangle \tag{39.4.7}$$

Since by taking the complex conjugation transpose of (39.3.1)[8]

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^* \tag{39.4.8}$$

and (39.4.7) becomes

$$\bar{\xi} = \langle\hat{\xi}\rangle = \langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\alpha^* + \alpha\right)\langle\alpha|\alpha\rangle = \sqrt{2}\Re e[\alpha] \neq 0 \tag{39.4.9}$$

Repeating the exercise for time-dependent case, when we let $\alpha \to \tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$, then, letting $\alpha = |\alpha|e^{-i\psi}$ yields

$$\langle\hat{\xi}(t)\rangle = \sqrt{2}|\alpha|\cos(\omega_0 t + \psi) \tag{39.4.10}$$

[8]Dirac notation is homomorphic with matrix algebra notation. $(\overline{\mathbf{a}} \cdot \mathbf{x})^\dagger = \mathbf{x}^\dagger \cdot (\overline{\mathbf{a}})^\dagger$.

By the same token,

$$\bar{\pi} = \langle \hat{\pi} \rangle = \langle \alpha | \hat{\pi} | \alpha \rangle = \frac{i}{\sqrt{2}} (\alpha^* - \alpha) \langle \alpha | \alpha \rangle = \sqrt{2} \Im m[\alpha] \neq 0 \tag{39.4.11}$$

For the time-dependent case, we let $\alpha \rightarrow \tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$,

$$\langle \hat{\pi}(t) \rangle = -\sqrt{2} |\alpha| \sin(\omega_0 t + \psi) \tag{39.4.12}$$

Hence, we see that the expectation values of the normalized coordinate and momentum just behave like a classical pendulum. There is however a marked difference: These values have standard deviations or variances that are non-zero. Thus, they have quantum fluctuation or quantum noise associated with them. Since the quantum pendulum is homomorphic with the oscillation of a quantum electromagnetic mode, the amplitude of a quantum electromagnetic mode will have a mean and a fluctuation as well.

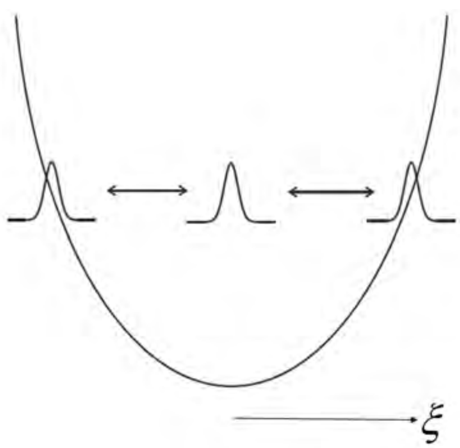

Fig. 3.4. A coherent-state wave function moves through the harmonic oscillator potential, between the classical turning points, without dispersion.

Figure 39.1: The time evolution of the coherent state. It follows the motion of a classical pendulum or harmonic oscillator (courtesy of Gerry and Knight [255]).

.

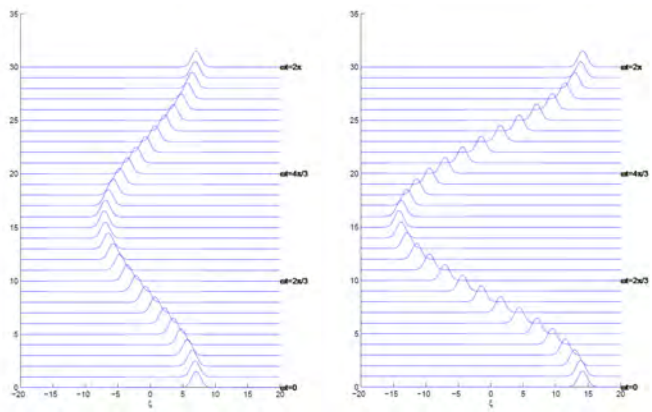

Figure 39.2: The time evolution of the coherent state for different α's. The left figure is for $\alpha = 5$ while the right figure is for $\alpha = 10$. Recall that $N = |\alpha|^2$.

.

39.4.1 Connecting Quantum Pendulum to Electromagnetic Oscillator

We see that the electromagnetic oscillator in a cavity is similar or homomorphic to a pendulum. The classical Hamiltonian is

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}\left[P^2(t) + Q^2(t)\right] = E \tag{39.4.13}$$

In the above, P is a normalized momentum and Q is a normalized coordinate, and their squares have the unit of energy. We have also shown that when the classical pendulum is elevated to be a quantum pendulum, then $H \to \hat{H}$, where $\hat{H} = \hbar\omega_l\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. Then Schrödinger equation becomes

$$\hat{H}|\psi, t\rangle = \hbar\omega_l\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\psi, t\rangle = i\hbar\partial_t|\psi, t\rangle \tag{39.4.14}$$

Our next task is to connect the electromagnetic oscillator to this pendulum. In general, the total energy or the Hamiltonian of an electromagnetic system is

$$H = \frac{1}{2}\int_V d\mathbf{r}\left[\varepsilon\mathbf{E}^2(\mathbf{r}, t) + \frac{1}{\mu}\mathbf{B}^2(\mathbf{r}, t)\right]. \tag{39.4.15}$$

It is customary to write this Hamiltonian in terms of scalar and vector potentials. For simplicity, we use a 1D cavity, and let $\mathbf{A} = \hat{x}A_x$, $\nabla \cdot \mathbf{A} = 0$ so that $\partial_x A_x = 0$, and letting $\Phi = 0$. Then $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\dot{\mathbf{A}}$, and the classical Hamiltonian from (39.4.15) for a

Maxwellian system becomes

$$H = \frac{1}{2}\int_V d\mathbf{r}\left[\varepsilon\dot{\mathbf{A}}^2(\mathbf{r},t) + \frac{1}{\mu}\left(\nabla\times\mathbf{A}(\mathbf{r},t)\right)^2\right]. \tag{39.4.16}$$

For the 1D case, the above implies that $B_y = \partial_z A_x$, and $E_x = -\partial_t A_x = -\dot{A}_x$. Hence, we let

$$A_x = A_0(t)\sin(k_l z) \tag{39.4.17}$$

$$E_x = -\dot{A}_0(t)\sin(k_l z) = E_0(t)\sin(k_l z) \tag{39.4.18}$$

$$B_y = k_l A_0(t)\cos(k_l z). \tag{39.4.19}$$

where $E_0(t) = -\dot{A}_0(t)$. After integrating over the volume such that $\int_V d\mathbf{r} = \mathcal{A}\int_0^L dz$, the Hamiltonian (39.4.16) then becomes

$$H = \frac{V_0\varepsilon}{4}\left(\dot{A}_0(t)\right)^2 + \frac{V_0}{4\mu}k_l^2 A_0^2(t). \tag{39.4.20}$$

where $V_0 = \mathcal{AL}$, is the mode volume. The form of (39.4.20) now resembles the pendulum Hamiltonian. We can think of $A_0(t)$ as being related to the displacement of the pendulum. Hence, the second term resembles the potential energy. The first term has the time derivative of $A_0(t)$, and hence, can be connected to the kinetic energy of the system. Therefore, we can rewrite the Hamiltonian as

$$H = \frac{1}{2}\left[P^2(t) + Q^2(t)\right] \tag{39.4.21}$$

where

$$P(t) = \sqrt{\frac{V_0\varepsilon}{2}}\dot{A}_0(t) = -\sqrt{\frac{V_0\varepsilon}{2}}E_0(t), \qquad Q(t) = \sqrt{\frac{V_0}{2\mu}}k_l A_0(t) \tag{39.4.22}$$

By elevating P and Q to be quantum operators,

$$P(t) \to \hat{P} = \sqrt{\hbar\omega_l}\hat{\pi}(t), \qquad Q(t) \to \hat{Q} = \sqrt{\hbar\omega_l}\hat{\xi}(t) \tag{39.4.23}$$

so that the quantum Hamiltonian now is

$$\hat{H} = \frac{1}{2}\left[\hat{P}^2 + \hat{Q}^2\right] = \frac{1}{2}\hbar\omega_l\left(\hat{\pi}^2 + \hat{\xi}^2\right) \tag{39.4.24}$$

similar to (39.1.3) as before except now that the resonant frequency of this mode is ω_l instead of ω_0 because these are the cavity modes, each of which is homomorphic to a quantum pendulum of frequency ω_l. An equation of motion for the state of the quantum system can be associated with the quantum Hamiltonian just as in the quantum pendulum case.

We have shown previously that

$$\hat{a}^\dagger + \hat{a} = \sqrt{2}\hat{\xi} \tag{39.4.25}$$

$$\hat{a}^\dagger - \hat{a} = -\sqrt{2}i\hat{\pi} \tag{39.4.26}$$

Then we can let

$$\hat{P} = -\sqrt{\frac{V_0\varepsilon}{2}}\hat{E}_0 = \sqrt{\hbar\omega_l}\hat{\pi} \tag{39.4.27}$$

Finally, we arrive at

$$\hat{E}_0 = -\sqrt{\frac{2\hbar\omega_l}{\varepsilon V_0}}\hat{\pi} = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{39.4.28}$$

Now that E_0 has been elevated to be a quantum operator $\hat{E}_0$, from (39.4.18), we can put in the space dependence to get

$$\hat{E}_x(z) = \hat{E}_0 \sin(k_l z) \tag{39.4.29}$$

Consequently,

$$\hat{E}_x(z) = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\hat{a}^\dagger - \hat{a}\right)\sin(k_l z) \tag{39.4.30}$$

Notice that in the above, $\hat{E}_0$, and $\hat{E}_x(z)$ are all Hermitian operators and they correspond to quantum observables that have randomness associated with them. Also, the operators are independent of time because they are in the Schrödinger picture.

In the Schrödinger picture, to get time dependence fields, one has to take the expectation value of the operators with respect to time-varying quantum state vector like the time-varying coherent state.

To let $\hat{E_x}$ have any meaning, it should act on a quantum state. For example,

$$|\psi_E\rangle = \hat{E_x}|\psi\rangle \tag{39.4.31}$$

Notice that thus far, all the operators derived are independent of time. To derive time dependence of these operators, one needs to find their expectation value with respect to time-dependent state vectors.[9]

To illustrate this, we can take expectation value of the quantum operator $\hat{E}_x(z)$ with respect to a time dependent state vector, like the time-dependent coherent state, Thus

$$\begin{aligned}\langle E_x(z,t)\rangle = \langle\alpha,t|\hat{E}_x(z)|\alpha,t\rangle &= \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\langle\alpha,t|\hat{a}^\dagger - \hat{a}|\alpha,t\rangle \\ &= \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\left(\tilde{\alpha}^*(t) - \tilde{\alpha}(t)\right)\langle\alpha,t|\alpha,t\rangle = -2\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}\Im m(\tilde{\alpha})\end{aligned} \tag{39.4.32}$$

Using the time-dependent $\tilde{\alpha}(t) = \alpha e^{-i\omega_l t} = |\alpha|e^{-i(\omega_l t + \psi)}$ in the above, we have

$$\langle E_x(z,t)\rangle = 2\sqrt{\frac{\hbar\omega_l}{\varepsilon V_0}}|\alpha|\sin(\omega_l t + \psi) \tag{39.4.33}$$

where $\tilde{\alpha}(t) = \alpha e^{-i\omega_l t}$. The expectation value of the operator with respect to a time-varying quantum state, in fact, gives rise to a time-varying quantity. The above, which is the average

[9]This is known as the Schrödinger picture.

of a random field, resembles a classical field. But since it is rooted in a random variable, it has a standard deviation in addition to having a mean.

We can also show that

$$\hat{B}_y(z) = k_l \hat{A}_0 \cos(k_l z) = \sqrt{\frac{2\mu\hbar\omega_l}{V_0}}\hat{\xi} = \sqrt{\frac{\mu\hbar\omega_l}{V_0}}(\hat{a}^\dagger + \hat{a}) \tag{39.4.34}$$

Again, these are time-independent operators in the Schrödinger picture. To get time-dependent quantities, we have to take the expectation value of the above operator with respect to to a time-varying quantum state.

39.5 Epilogue

In conclusion, the quantum theory of light is a rather complex subject. It cannot be taught in just two lectures, but what we wish is to give you a peek into this theory. It takes much longer to learn this subject well: after all, it is the by product of a century of intellectual exercise. This knowledge is still very much in its infancy. Hopefully, the more we teach this subject, the better we can articulate, understand, and explain this subject. When James Clerk Maxwell completed the theory of electromagnetics over 150 years ago, and wrote a book on the topic, rumor has it that most people could not read beyond the first 50 pages of his book [256]. But after over a century and a half of regurgitation, we can now teach the subject to undergraduate students! When Maxwell put his final stroke to the equations named after him, he could never have foreseen that these equations are valid from nano-meter lengthscales to galactic lengthscales, from static to ultra-violet frequencies. Now, these equations are even valid from classical to the quantum world as well!

Hopefully, by introducing these frontier knowledge in this course, it will pique your interest enough in electromagnetic field theory, so that you will take this as a life-long learning experience.